Received: *March 31, 2000*, Accepted: *April 27, 2000*

On inherited duality in $\mathcal{N}=1$ $d=4$ supersymmetric gauge theories

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Abstract: Four-dimensional $\mathcal{N}=1$ supersymmetric gauge theories with two adjoints and a quartic superpotential are believed, from AdS/CFT duality, to have $\mathrm{SL}(2,\mathbb{Z})$ invariance. In this note we review an old, unpublished argument for this property, based solely on field theory. The technique involves a complexified flavor rotation which deforms an $\mathcal{N}=2$ supersymmetric gauge theory with matter to an $\mathcal{N}=1$ theory, leaving all holomorphic invariants unchanged. We apply this to the $\mathcal{N}=1$ gauge theory with two massless adjoints and show that it has the same auxiliary torus as that of $\mathcal{N}=4$ gauge theory, from which $\mathrm{SL}(2,\mathbb{Z})$ invariance follows. In the appendix, we check that our arguments are consistent with earlier work on the SU(2) case. Our technique is general and applies to many other $\mathcal{N}=1$ theories.

Keywords: Nonperturbative Effects, Duality in Gauge Field Theories, Supersymmetry and Duality.

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Recently, the theory of four-dimensional $\mathcal{N} = 1$ supersymmetric gauge theory with two chiral superfields in the adjoint representation has received considerable attention. In particular, its $\mathrm{SL}(2,\mathbb{Z})$ electromagnetic self-duality and its AdS_5 dual representation have been discussed in numerous papers [1]. Similar theories were considered in [2]. For large gauge groups, $\mathrm{SL}(2,\mathbb{Z})$ invariance for this theory has been strongly suggested using its conjectured duality with type IIB supergravity, which has a semiclassical symmetry of this type. Since IIB string theory has $\mathrm{SL}(2,\mathbb{Z})$ as well, then, if one accepts the AdS duality for arbitrary gauge and 't Hooft couplings, the $\mathrm{SL}(2,\mathbb{Z})$ duality of the field theory should extend to any gauge group.

The $\mathrm{SL}(2,\mathbb{Z})$ duality has never been established in the literature from purely field theoretic considerations. However, there is an unpublished argument in favor of this symmetry, and in view of recent interest in this theory, it seems appropriate to make it more widely known. Specifically, we want to consider, for any gauge group G, the $\mathcal{N} = 1$ supersymmetric gauge theory with two chiral superfields ϕ_1, ϕ_2 in the adjoint representation and a superpotential

$$W = h\,\mathrm{tr}[\phi_1, \phi_2][\phi_1, \phi_2]\,. \tag{1}$$

The theory is non renormalizable and must be defined with a cutoff. However we are interested in its infrared behavior, where it flows to a conformal field theory. The coupling h, although canonically of inverse mass dimension, becomes dimensionless in the infrared. Define h_* to be its value in the infrared, in some suitable scheme. In [3] it was shown that there is a continuous set of conformal field theories near $h_* = 0$. In other words, the quartic superpotential above is an exactly marginal perturbation of the interacting conformal field theory with zero superpotential. The marginal coupling h_* is inherited from the marginal gauge coupling τ of $\mathcal{N} = 4$ supersymmetric Yang-Mills theory, and in [3] it was conjectured that the $\mathrm{SL}(2,\mathbb{Z})$ self-duality that acts on τ is also inherited by the theory with two adjoints.

In this paper we claim the following. First, we verify that there exists a space of conformal field theories which we label by the parameter ρ, or equivalently $q = e^{2\pi i\rho}$, of which h_* is a non-trivial and presumably scheme-dependent function. Second, we claim that the set of theories parameterized by q, or equivalently $h_*(\rho)$, has $\mathrm{SL}(2,\mathbb{Z})$ duality; the conformal theory with parameter ρ, and the theories with parameter $(a\rho + b)/(c\rho + d)$, a, b, c, d integers, $ad - bc = 1$, are actually different descriptions of the same theory. In particular, as seen in the case of SU(2) studied in [4], the theory with $h_* = 0$ is equivalent to a theory with a particular non-zero value of h_*.

To prove this we employ a complexified flavor symmetry transformation, under which all holomorphic quantities are invariant even though the theory as a whole is altered. One subtlety, not resolved here, is how precisely to match the dimensionless coupling τ of the $\mathcal{N} = 4$ theory on to the coupling $h_*(\rho)$ of the two-adjoint theory. This issue need not be settled for the $\mathrm{SL}(2,\mathbb{Z})$ invariance to be established.

JHEP04(2000)029

Let us begin by reviewing the arguments of [3] concerning the theory with two adjoints. Consider an $\mathcal{N}=1$ supersymmetric field theory in four dimensions, with a gauge group G and N_f chiral multiplets ϕ_i, $i=1,\ldots,N_f$ in the adjoint of G. If $N_f=0$ the theory is $\mathcal{N}=1$ Yang-Mills and shows confinement and chiral symmetry breaking. If $N_f=1$ and the superpotential is zero, the theory is $\mathcal{N}=2$ supersymmetric and has a Coulomb branch with special points where magnetically charged BPS states become massless [5]. If $N_f=3$ and the superpotential vanishes then the theory is infrared free. However, with a renormalizable superpotential

$$y\,\mathrm{tr}[\phi_1,\phi_2]\phi_3\,,$$

the theory will flow to a non-trivial fixed point, becoming $\mathcal{N}=4$ supersymmetric in the far infrared. If $y=\sqrt{2}$, then the theory is strictly $\mathcal{N}=4$ supersymmetric and is conformal at all scales. There is a one-complex-dimensional space of such theories, indexed by the gauge coupling and theta angle through the exactly marginal parameter $\tau=\theta/2\pi+4\pi i/g^2$.

For $N_f=2$, a simple argument shows that with the non-renormalizable superpotential

$$W=h\,\mathrm{tr}[\phi_1,\phi_2][\phi_1,\phi_2]\,,$$

the theory is expected to flow to a point on a one-complex-dimensional space of conformal fixed points. Specifically, the requirement that the beta functions for h and for the gauge coupling g both vanish reduces to a single condition on the two couplings [3]. To see this, note that the anomalous mass dimensions of ϕ_1 and ϕ_2 are equal by symmetry, so the beta functions take the form

$$\beta_g=-f(g)[C_2(G)+2C_2(G)\gamma_\phi]\,;\qquad \beta_h=h[1+2\gamma_\phi]\,,\tag{2}$$

where these formulas are exact as a consequence of $\mathcal{N}=1$ non-renormalization theorems. (Here $C_2(G)$ is the second Casimir of the adjoint representation.) Only one condition, namely $\gamma_\phi(g,h)=-1/2$, is required for the two beta functions to vanish; therefore, if there are any solutions to this condition, they will typically form one-complex-dimensional subspaces in the two-complex-dimensional space of couplings.[1]

Unfortunately, the condition $\gamma_\phi(g,h)=-1/2$ can only be satisfied well outside the realm of perturbation theory, and it cannot be proven without a shadow of a

[1] If G has a three-index symmetric invariant, we may consider for $N_f=3$ the superpotential $y\,\mathrm{tr}[\phi_1,\phi_2]\phi_3+y^s_{ijk}\,\mathrm{tr}\{\phi_i,\phi_j\}\phi_k$. If $y^s\neq 0$, the theory may still be conformal [6] but has only $\mathcal{N}=1$ supersymmetry; the space of conformal theories is three-complex-dimensional (including the $\mathcal{N}=4$ subspace). Similarly one may add $h_2\,\mathrm{tr}\{\phi_1,\phi_2\}\{\phi_1,\phi_2\}+h_3[\mathrm{tr}\{\phi_1,\phi_1\}\{\phi_1,\phi_1\}+\mathrm{tr}\{\phi_2,\phi_2\}\{\phi_2,\phi_2\}]$ to the superpotential of the theory with $N_f=2$. Since the new β functions $\beta_{h_2}=h_2[1+2\gamma_\phi]$; $\beta_{h_3}=h_3[1+2\gamma_\phi]$ are proportional to the other two, the vanishing of all four beta functions gives only one condition on four couplings, $\gamma_\phi(g,h,h_2,h_3)=-1/2$, so again we find a three-complex-dimensional space of fixed points. More details are found in [3].

doubt that solutions exist. However, there are strong reasons to believe that they are present. (Recent large-N results [1] support this point of view, of course.) We will assume for the remainder of this paper that there is a unique and connected space of solutions to this equation, and that this space contains a single point with $h = 0$ and g equal to some special value g_*.

We will now focus our attention on the space of conformal theories with $\gamma_\phi(g,h) = -1/2$. This one-complex-dimensional space has a single marginal coupling as its parameter. These theories are particularly interesting as they can be reached through a simple deformation of $\mathcal{N} = 4$ Yang-Mills. In particular, consider $\mathcal{N} = 4$ Yang-Mills, with gauge coupling τ, deformed by a mass term.

$$W = \sqrt{2}\,\mathrm{tr}[\phi_1, \phi_2]\phi_3 + \frac{1}{2} m_3\, \mathrm{tr}\, \phi_3^2.$$

(In the following we will be careful to distinguish τ, the coupling of the high-energy $\mathcal{N} = 4$ theory, from both the running gauge coupling g and the exactly marginal parameter of the low-energy $\mathcal{N} = 1$ conformal theories, which we will call ρ.) At scales below m_3 (more precisely, below some physical scale related nontrivially to the holomorphic parameter m_3) we may integrate out ϕ_3. The theory becomes

$$W = -\frac{1}{m_3}\, \mathrm{tr}[\phi_1, \phi_2][\phi_1, \phi_2]\,.$$

At the cross-over scale m_3 the theory is usefully parameterized by the gauge coupling $g = 1/\sqrt{\mathrm{Im}\,\tau}$ and $h \sim -1/m_3$. At low energy it flows to a fixed point with couplings $g_*(\tau)$ and $h_*(\tau)$; the initial gauge coupling of the $\mathcal{N} = 4$ theory specifies which $\mathcal{N} = 1$ conformal field theory will be reached in the infrared. Note that for $\tau \to i\infty$, $h_* \to 0$ but g_* is finite.

There is a subtlety involved in matching the theory above the scale m_3 with that below m_3. The symmetries of the theory permit non-perturbative corrections to the superpotential, making it of the form

$$W = h_{\mathrm{eff}}\, \mathrm{tr}[\phi_1, \phi_2][\phi_1, \phi_2]\,, \tag{3}$$

where

$$h_{\mathrm{eff}} = -\frac{f(q)}{m_3}, \quad f(q) = 1 + \mathcal{O}(q)\,.$$

The behavior of $f(q)$ for small q (large $\mathrm{Im}\,\tau$), the weak coupling region, is determined by perturbation theory. The only direct constraint on f is that it be single-valued under shifts of the theta angle by 2π, that is $q \to e^{2\pi i} q$. Higher order corrections to $f(q)$ are associated with instantons, and have not been determined beyond low orders. A similar ambiguity arises if we attempt to exchange the two parameters (τ, m_3) for $(h_{\mathrm{eff}}, \Lambda)$, where Λ is the holomorphic dynamical scale of the low-energy asymptotically-free gauge theory. By symmetry,

$$\Lambda^{C_2(G)} \sim m_3^{C_2(G)}\, g(q), \quad g(q) = q + \mathcal{O}(q^2)\,. \tag{4}$$

Again, only the leading behavior of $g(q)$ at weak coupling is determined, and consequently we will not use Λ in most of our discussion. These ambiguities will not affect our general arguments.

The $\mathcal{N}=4$ theory has a duality symmetry, namely an $\mathrm{SL}(2,\mathbb{Z})$ symmetry generated by the semiclassical symmetry $\tau \to \tau+1$ and the strong-weak coupling transformation $\tau \to -1/\tau$. This means the space of $\mathcal{N}=4$ theories is smaller than it appears due to discrete identifications. Since it seems that there is a one-to-one map between the space of $\mathcal{N}=4$ theories and the infrared two-adjoint conformal field theories, it is natural to guess [3] that the space of two-adjoint conformal theories is also reduced by the same discrete identifications, taking the form of $\mathrm{SL}(2,\mathbb{Z})$ transformations on the low-energy marginal coupling constant ρ. We will call this the "inherited duality" conjecture.

We now prove this conjecture, using the following trick. First, consider the $\mathcal{N}=4$ theory with *two* mass deformations.

$$W=\sqrt{2}\,\mathrm{tr}[\phi_1,\phi_2]\phi_3+\frac{1}{2}m_2\,\mathrm{tr}\,\phi_2^2+\frac{1}{2}m_3\,\mathrm{tr}\,\phi_3^2\,.$$

If $m_2=-m_3=\hat{m}$ the theory is $\mathcal{N}=2$ supersymmetric; it is the theory of $\mathcal{N}=2$ Yang-Mills with a massive adjoint hypermultiplet, first studied in [7, 8]. The moduli space is a Coulomb branch with an auxiliary Seiberg-Witten torus and Seiberg-Witten form, as in [5]; the torus can be used to specify the low-energy gauge couplings, and together with the form gives the low-energy Kähler potential (the effective lagrangian up to second order in momentum). The torus is a function of the holomorphic quantities $\hat{m}^2=-m_2m_3$ and q, and of the holomorphic coordinates on the moduli space; the low-energy gauge coupling is also a holomorphic function of these quantities. For $\hat{m}=0$ the theory is $\mathcal{N}=4$ supersymmetric and the auxiliary torus is invariant under $\mathrm{SL}(2,\mathbb{Z})$, as expected; for $\hat{m}\neq 0$ the torus is $\mathrm{SL}(2,\mathbb{Z})$ *covariant*, although it can be written in an $\mathrm{SL}(2,\mathbb{Z})$ invariant form with a suitable choice of coordinates on the moduli space [7, 8].

Now consider the global symmetries of this model. For $\hat{m}=0$ there is an $\mathrm{SO}(6)$ R-symmetry containing an $\mathrm{SU}(3)$ flavor symmetry under which ϕ_i are triplets. Let us consider its $\mathrm{SU}(2)$ subgroup which acts on ϕ_2,ϕ_3 only. Mass terms $m^{ij}\phi_i\phi_j$, $i,j=2,3$ break the symmetry; m^{ij} transforms in the $\mathbf{3}$ of $\mathrm{SU}(2)$. The $\mathcal{N}=2$ case just discussed requires $m^{ij}=\hat{m}(\sigma^3)^{ij}$, where σ^a are the Pauli matrices. A transformation

$$\phi_2 \longrightarrow e^{i\alpha}\phi_2\,,\qquad \phi_3 \longrightarrow e^{-i\alpha}\phi_3\,,$$

where α is real, is an $\mathrm{SU}(2)$ flavor-symmetry transformation which changes the phases of m_2,m_3 but leaves the theory invariant.

The observation which permits a proof of the inheritance conjecture involves the fact that holomorphic quantities, such as the low-energy gauge coupling, can only depend on the holomorphic product $\det m = m_2 m_3$. Nothing holomorphic depends on m_2/m_3. Consider the behavior of the theory under the transformation

$$\phi_2 \longrightarrow e^{\alpha}\phi_2\,, \qquad \phi_3 \longrightarrow e^{-\alpha}\phi_3\,, \tag{5}$$

where again α is real. This imaginary SU(2) transformation is not a symmetry. $\mathcal{N}=2$ supersymmetry is broken to $\mathcal{N}=1$, and the Kähler potential, which no longer need depend only on holomorphic quantities, is altered. However, *although the theory as a whole is not invariant under this transformation, all holomorphic quantities are unchanged.* In particular, $\det m$ is invariant under this transformation, and so the torus, and the low-energy gauge couplings, are unaffected. It follows, therefore, that for $\det m$ non zero the low-energy theory inherits the torus of the $\mathcal{N}=2$ theory with $m_2 = -m_3 = \sqrt{-\det m}$.

To obtain the theory with two massless adjoints and a quartic superpotential, we take $m_3 \neq 0$ but $m_2 m_3 = 0$. The torus of this theory is the same as for $m_2 = m_3 = 0$, the $\mathcal{N}=4$ case; it is invariant under $\mathrm{SL}(2,\mathbb{Z})$. At all points on the moduli space where the low-energy gauge group is abelian, all infrared gauge couplings τ_L are equal to the ultraviolet coupling τ. At the origin of moduli space, the torus has no massive parameters, indicating a set of conformal theories indexed by τ, with discrete identifications under $\mathrm{SL}(2,\mathbb{Z})$. This indicates a one-to-one map between the marginal parameter ρ and the ultraviolet coupling τ, so we may take $\rho = \tau$. Thus, the set of conformal field theories with group G and two adjoint chiral multiplets inherits $\mathrm{SL}(2,\mathbb{Z})$ duality from its $\mathcal{N}=4$ parent.

It may seem strange at first that the deformation by a mass term $\frac{1}{2} m_3 \operatorname{tr}\phi_3^2$ does not change the low-energy torus and its attendant gauge couplings. However, there is a simple physical explanation, most easily presented in the case of $G = \mathrm{SU}(2)$. In this case the theory has a moduli space with a single coordinate $u = \frac{1}{2}\operatorname{tr}(\phi_1^2)$. For given $\langle u \rangle$, the gauge group is broken to U(1) and the low-energy gauge coupling τ_L is the modular parameter of a holomorphic torus [7], reproduced in eq. (8), which is a function of $\hat{m}^2$, u, and τ. The imaginary global symmetry transformation eq. (5) breaks the supersymmetry to $\mathcal{N}=1$, but the gauge coupling is still the modular parameter of a holomorphic curve [9]. Since the complexified flavor rotation eq. (5) leaves $\det m$, u and τ invariant, this curve is the same as that of the $\mathcal{N}=2$ theory with a massive adjoint hypermultiplet, except that $\hat{m}^2$ is replaced with $-m_2 m_3$ in eq. (8). But how can it make physical sense that the low-energy U(1) coupling τ_L should not depend in any way on m_2/m_3? This is quite easy to see in the weak-coupling limit. Let us take the expectation value of u to be large compared with m_2 and m_3, so that perturbation theory is valid. The SU(2) group is broken to U(1) at the scale $\sqrt{u}$; the fields ϕ_2, ϕ_3 are massive charged fields, with mass matrix

$$\begin{bmatrix} m_2 & 2\sqrt{u} \\ -2\sqrt{u} & m_3 \end{bmatrix}\,. \tag{6}$$

Referring to their masses as μ_2 and μ_3, we note $\mu_2\mu_3 = 4u + m_2m_3$. Consider the case where $|\mu_3| > |2\sqrt{u}| > |\mu_2|$. (All other cases lead to the same result, though the description of the physics will be different.) Above both masses, the coupling does not run. Between μ_3 and $2\sqrt{u}$, the SU(2) theory has only two adjoints and runs toward strong coupling, generating a logarithm of $\mu_3/2\sqrt{u}$ with beta-function coefficient -2. At the scale $2\sqrt{u}$ the gauge group is broken to U(1), and the remaining charged fields cause the coupling to run toward weak coupling, generating a logarithm of $2\sqrt{u}/\mu_2$ with beta-function coefficient $+2$. Below the scale μ_2 the coupling ceases to run, and the low-energy coupling constant is

$$\tau_L = \tau + \frac{1}{i\pi}\log\left[\frac{\mu_2\mu_3}{4u}\right], \tag{7}$$

which depends only on u and on m_2m_3. Evidently, the effects of raising m_3 and lowering m_2 are being arranged to cancel. Remarkably, this perturbative cancellation generalizes to the full non-perturbative behavior of the holomorphic properties of the theory.

So far we have learned that varying some massive parameters in the theory of SU(2) with three adjoints can change the theory at non-zero momenta (influencing the Kähler potential and massive states) while leaving the holomorphic part of the far-infrared physics the same. In particular, $SL(2,\mathbb{Z})$ invariance is preserved. But now we may consider the limit where $|m_3| \gg |u|, |m_2|$. Below m_3 we have the theory with superpotential eq. (1), along with a mass for ϕ_2. For small q and $|qm_3^2| \ll |u| \ll |m_3^2|$, the gauge coupling is weak everywhere and the analysis involving eqs. (6) and (7) still applies. In the limit $m_2 \ll u/m_3$, we have $\mu_3 \approx m_3$, $\mu_2 \approx 4u/m_3$, with $\mu_2\mu_3 = 4u$; the coupling runs with a beta function of -2 between m_3 and $2\sqrt{u}$, then with beta function $+2$ between $2\sqrt{u}$ and $2u/m_3$. Thus, in the $m_2 \to 0$ limit, all running effects above and below $2\sqrt{|u|}$ cancel precisely in eq. (7), and we find that the gauge coupling τ_L of the low-energy U(1) theory is the same as that of the high-energy theory, τ, just as in the unbroken $\mathcal{N} = 4$ gauge theory. This reflects the fact that the torus depends in this limit only on q and u, and not on m_3. (Another example of this type appears in [10].)

As in the $\mathcal{N} = 4$ theory, the limit $u \to 0$ leaves the torus a scale-invariant $SL(2,\mathbb{Z})$-invariant function of the parameter $q = e^{2\pi i\tau}$. We take this as evidence that at $u = m_2 = 0$ the theory eq. (1) flows to a non-trivial conformally invariant theory in the infrared. Indeed there is a set of conformal field theories indexed by q, in agreement with [3] and eq. (2).

The full picture for the SU(2) case is now the following. The $\mathcal{N} = 4$ conformal theories are parameterized by τ, or equivalently $q = e^{2\pi i\tau}$ living on the disk $|q| \leq 1$. These theories are identified under $SL(2,\mathbb{Z})$ transformations. In the $\mathcal{N} = 1$ conformal theories with two adjoints and superpotential eq. (1), there is also a marginal parameter ρ, which we may choose to be numerically equal to τ, such that $q = e^{2\pi i\rho}$

has the same properties as q in the $\mathcal{N}=4$ theory. The limit $q=0$ corresponds to $h=0$, but the general relation between h and q is non trivial and presumably scheme-dependent. The transformations $\rho \to -1/\rho$ and $\rho \to \rho+1$ correspond to electric-magnetic and magnetic-dyonic dualities. In the appendix we show, using the integrating-in techniques of [11], that these claims agree with those described in [4].

There are many other interesting $\mathcal{N}=1$ theories which will also have large discrete invariance groups. The simplest can be generated by taking the theory of $\mathcal{N}=2$ SU(2) with four hypermultiplets. This theory was shown by Seiberg and Witten to have SL$(2,\mathbb{Z})$ invariance as well [7]. The flavor symmetry of the doublets is SU(8), which is broken by the $\mathcal{N}=2$ superpotential term to SO(8). By giving masses to some of the quarks and performing complexified SO(8) transformations, we can again find many theories with non-renormalizable operators which are exactly marginal in the infrared, and whose couplings transform under SL$(2,\mathbb{Z})$. Examples of such theories appear in [12, 13].

Another related theory is $\mathcal{N}=2$ supersymmetric SU$(N)\times$SU(N) with two hypermultiplets in the $(\mathbf{N},\bar{\mathbf{N}})$ representation. This has a duality group given by the symmetry group of a torus with two identical marked points [14]. The superpotential $W=\Phi_1(Q_1\tilde{Q}_1+Q_2\tilde{Q}_2)+\Phi_2(\tilde{Q}_1Q_1+\tilde{Q}_2Q_2)$ has a U(2) flavor symmetry, broken by masses $m^{ij}Q_i\tilde{Q}_j$, under which m is a triplet plus a singlet. If equal and opposite masses for the hypermultiplets are added, a complexified flavor rotation can leave the theory with mass terms $mQ_1\tilde{Q}_2+m'Q_2\tilde{Q}_1$. With m' finite and $m=0$ we obtain a set of conformal theories, with a quartic superpotential involving $\Phi_1,\Phi_2,Q_1,\tilde{Q}_2$, that inherits the duality group of its parent.

Our technique is a simple and powerful tool for showing that large duality groups are widespread in $\mathcal{N}=1$ supersymmetry. It would be interesting to understand more deeply the mathematical underpinning of these results, and to find a brane-based realization of our method.

Acknowledgments

We would like to thank N. Dorey, N. Seiberg and E. Witten for discussions. K.I. and M.J.S. were supported by National Science Foundation grant NSF PHY95-13835 and by the W.M. Keck Foundation. K.I. was also supported by UCSD grant DOE-FG03-97ER40546. P.C.A. was supported by National Science Foundation grants NSF PHY94-07194 and NSF PHY95-13717 and by an A.P. Sloan Foundation fellowship. This work was done mostly at Rutgers University, during which time M.J.S., P.C.A. and R.G.L were supported by Department of Energy contract DE-FG05-90ER40559.

Appendix

In this appendix we show that the application of our techniques to the SU(2) theory with two adjoints reproduces the results of [4] for SO(3) with two triplets.

Consider SU(2) $\mathcal{N}=4$ Yang-Mills with gauge coupling τ deformed by $\mathcal{N}=1$-preserving masses, giving the superpotential

$$W=\sqrt{2}\,\mathrm{tr}\,\phi_1[\phi_2,\phi_3]+m^{ij}u_{ij}\,,$$

where classically $u_{ij}=1/2\ \mathrm{tr}(\phi_i\phi_j)$. Denote the three eigenvalues of m^{ij} by m_i, and define $u\equiv u_{11}$.

Our complexified flavor rotation trick implies that for $m_1=0$, the low-energy effective coupling τ_L on the Coulomb branch of this theory is equal to that of the SU(2) $\mathcal{N}=2$ theory with a fundamental hypermultiplet of mass $\hat{m}^2=-m_2m_3$. In [7], τ_L is given as the modular parameter of the auxiliary torus

$$y^2=\prod_{i=1}^{3}\left(x-e_i(q)\tilde{u}+\frac{1}{4}e_i^2(q)m_2m_3\right)\,, \tag{8}$$

where

$$\tilde{u}\equiv\langle u\rangle+(1/8)e_1(q)m_2m_3\,, \tag{9}$$

$q\equiv e^{2\pi i\tau}$, and the $e_i(q)$ are the usual modular forms associated with the torus, satisfying $e_1+e_2+e_3=0$, $e_1-e_2=[\theta_3(\tau)]^4$, etc. with small-$q$ expansions

$$\begin{aligned}
e_1(q)&=\frac{2}{3}+16q+\mathcal{O}\left(q^2\right)\,,\\
e_2(q)&=-\frac{1}{3}-8q^{1/2}-8q+\mathcal{O}\left(q^{3/2}\right)\,,\\
e_3(q)&=-\frac{1}{3}+8q^{1/2}-8q+\mathcal{O}\left(q^{3/2}\right)\,.
\end{aligned} \tag{10}$$

For fixed m_2m_3 and $\tilde{u}$ the torus is $\mathrm{SL}(2,\mathbb{Z})$ invariant. This follows from the modular properties of the e_i, which are interchanged with one another under $\mathrm{SL}(2,\mathbb{Z})$. In particular, $e_1\leftrightarrow e_2$ under $\tau\to-1/\tau$, while $e_2\leftrightarrow e_3$ under $\tau\to\tau+1$. Note that with the definition of $\tilde{u}$ given in eq. (9), the Coulomb branch coordinate $\langle u\rangle$ transforms under $\mathrm{SL}(2,\mathbb{Z})$.

There is, however, an ambiguity involved in the definitions of u and the masses. The symmetries of the theory permit non-perturbative redefinitions of the form

$$u\longrightarrow k_1(q)u+q\,k_2(q)m_2m_3\quad k_i(q)=1+\mathcal{O}(q)\,. \tag{11}$$

and

$$m_i\longrightarrow \ell_i(q)m_i\,,\quad \ell_i(q)=1+\mathcal{O}(q)\,. \tag{12}$$

The reason is that u and the bare masses are only defined in the weak coupling ($q\to 0$) limit, which the above redefinitions preserve. (More generally, $u_{ij}\approx 1/2\,\mathrm{tr}(\phi_i\phi_j)$ and m^{ij} can suffer such redefinitions, with u_{ij} mixing at order q with the subdeterminants of m^{ij}.)

It is convenient to use the above freedom to redefine the masses by

$$m_2 m_3 \longrightarrow 9 e_2(q) e_3(q) m_2 m_3 \,, \tag{13}$$

which can be checked to be of the form of eq. (12) using eq. (10). The virtue of this redefinition is that, when used in eq. (9), $\langle u \rangle$ is $\mathrm{SL}(2,\mathbb{Z})$ invariant. This is convenient for keeping manifest $\mathrm{SL}(2,\mathbb{Z})$ invariance in our calculations when we integrate u out, as we will do shortly.

We can now recover the results of [4] for the SO(3) theory with two triplets and no superpotential. This is the limit of our theory in which $m_3 \to \infty$, $q \to 0$ keeping $\Lambda \sim m_3 \sqrt{q}$ fixed. In [4] three descriptions of the theory were found, in terms of electric, magnetic, and dyonic states. The electric description has no superpotential; the magnetic (dyonic) description (upon integrating out some massive singlets used in the presentation of [4]) has superpotential

$$W = -\frac{\eta}{8\Lambda} \det_{j,k} \left[\mathrm{tr}(\phi_j \phi_k) \right] \,, \tag{14}$$

where $\eta = 1$ (-1) in the magnetic (dyonic) description.

Now, the SU(2) theory with one massless and two massive adjoints has a superpotential which enforces the relation (8), with the substitution of eq. (13), by way of a Lagrange multiplier λ

$$W = \lambda \left[y^2 - \prod_{r=1}^{3} \left(x - e_r \left[u + \frac{9}{8} e_1 e_2 e_3 m_2 m_3 \right] + \frac{1}{4} e_r^2 m_2 m_3 \right) \right] .$$

Adding a mass for ϕ_1 takes $W \to W + m_1 u$. Upon integrating u out, one finds a low energy superpotential with three branches

$$W_{r,L} = m_1 u_r(q, \hat{m})|_{dW/du} = 0 = -\frac{1}{8} m_1 m_2 m_3 \Big[9 e_1(q) e_2(q) e_3(q) + 2 e_r(q) \Big] \,, \tag{15}$$

where u_r is one of the three values of u at which the torus (8) becomes singular. (See also [15].)

Taking $m_1 m_2 = \det \widetilde{m}$ where $\widetilde{m}^{jk}$ for $j,k = 1,2$ is the mass matrix for $\phi_{1,2}$, and integrating the two adjoint fields ϕ_1 and ϕ_2 back in, as in [11], gives

$$W_r = \left[W_{r,L} - \widetilde{m}^{jk} u_{jk} \right]_{dW/d\tilde{m}=0} = \frac{2}{m_3 [9 e_1 e_2 e_3 + 2 e_r]} \det_{j,k=1,2} u_{jk}$$

The $q \to 0$, $m_3 \to \infty$ limit, for which $u_{ij} \to \frac{1}{2} \mathrm{tr}(\phi_i \phi_j)$, gives

$$\begin{aligned}
W_1 &= \frac{4 \det_{j,k} u_{jk}}{m_3 e_1 (9 e_2 e_3 + 2)} \longrightarrow 0 \,, \\
W_2 &= 4 \frac{\det_{j,k} u_{jk}}{m_3 e_2 (9 e_1 e_3 + 2)} \longrightarrow -\frac{\det_{j,k} [\mathrm{tr}(\phi_j \phi_k)]}{8\Lambda} \,, \\
W_3 &= \frac{4 \det_{j,k} u_{jk}}{m_3 e_3 (9 e_1 e_2 + 2)} \longrightarrow \frac{\det_{j,k} [\mathrm{tr}(\phi_j \phi_k)]}{8\Lambda} \,,
\end{aligned}$$

matching to the electric, magnetic, and dyonic superpotentials of eq. (14), respectively. In this way we see explicity how the $\mathrm{SL}(2,\mathbb{Z})$ duality transformations $\tau \to -1/\tau$ and $\tau \to \tau + 1$ correspond to electric-magnetic duality and magnetic-dyonic duality. A Γ_2 subgroup of $\mathrm{SL}(2,\mathbb{Z})$ leaves the descriptions invariant, while $\mathrm{SL}(2,\mathbb{Z})/\Gamma_2 \cong S_3$ permutes the three descriptions. These facts were already understood in [4] from the connection of the theory with the duality of $\mathcal{N}=4$ SU(2) gauge theory; here we understand it as following from the $\mathrm{SL}(2,\mathbb{Z})$ duality of the $\mathcal{N}=1$ theory itself. Furthermore, it is easy to check that redefining the fields and parameters by putting in arbitrary $g(q)$, $k_i(q)$, and $\ell_i(q)$ [see eqs. (4), (11) and (12)] has no effect on the above calculation.

One could also attempt to start from the low energy superpotentials in eq. (15) and integrate back in all three ϕ_j; we then expect to recover the $\mathcal{N}=4$ theory with coupling q. However, this is more difficult than in the above two-flavor case because we must introduce not only $\mathrm{tr}(\phi_i\phi_j)$ but also the gauge invariant non-quadratic operator $\det(\phi)$. (Here ϕ is a 3×3 matrix in flavor and color.) So we instead consider going in the other direction, attempting to recover eq. (14) in the theory with two adjoints by integrating massive ϕ_3 out of the $\mathcal{N}=4$ theory.

Consider the theory with three adjoints, superpotential $W = \sqrt{2}\beta \det\phi$, and coupling q. For $\beta = 1$ the theory is conformal. For $\beta \neq 1$ the theory has only $\mathcal{N}=1$ supersymmetry, but flows until it reaches the IR attractive $\mathcal{N}=4$ fixed point where the physical (nonholomorphic) coupling $\beta = 1$. As discussed in [4], the quantity $t \equiv \beta^4 q$ is invariant under this RG flow (more generally, $\beta^{2C_2(G)}q$ is invariant), and the low-energy $\mathcal{N}=4$ conformal theory has coupling t. The theory is nowhere weakly coupled unless $t \ll 1$.

Adding $\frac{1}{2}m_3 u_{33}$ to the superpotential and integrating out ϕ_3, we find that symmetries ensure that the low-energy superpotential is

$$W_L = \frac{\beta^2 s(t)}{m_3} \det_{i,j=1,2} \mathrm{tr}(\phi_i\phi_j)\,, \tag{16}$$

where $s(t)$ is an unknown function. (The symmetries ensure that in this case u_{ij} and $\frac{1}{2}\mathrm{tr}(\phi_i\phi_j)$ are proportional, differing by another function of t.) For β fixed and $q \to 0$, the theory is weakly coupled and we may integrate out ϕ_3 classically, which reveals that $s(t=0)=1$. At finite t, $s(t)$ is undetermined. However, we know the $\beta = 1, q \to 0$ theory is $\mathrm{SL}(2,\mathbb{Z})$-dual to $\beta = 1, q \to 1$ and $\beta = 1, q \to e^{2\pi i}$, which are the magnetic and dyonic descriptions. In these descriptions, where $t \sim 1$, the classical analysis is not valid, and $s(t)$ may differ from 1. We now determine $s(1)$ in the scheme used in [4].

Taking $\beta = 1$, $q \to 0$, $m_3 \to \infty$, with $\Lambda = m_3 q^{1/2}$ held fixed, eq. (16) obviously yields the expected electric low-energy superpotential, eq. (14) with $\eta = 0$. A magnetic description of the same theory should be obtained by studying the theory with $\beta = 1$ and $q \to 1$, but it is convenient instead to study a theory with the same *infrared* physics, namely one with $\beta = q^{-1/4}$ and $q \to 0$; both theories have $t = 1$. The latter theory can be defined by holding the strong coupling scale $\Lambda \equiv m_3 q^{1/2}$ fixed, as in [4]. From eq. (16) this limit has superpotential

$$W_L = \frac{s(1)}{m_3 q^{1/2}} \det_{i,j=1,2} \mathrm{tr}(\phi_i \phi_j)\,,$$

which agrees with eq. (14) for $\eta = 1$ provided $s(1) = 1/8$. The dyonic description is given by taking $q \to e^{2\pi i}$, which changes the sign of the superpotential in agreement with eq. (14).

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